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Abstract

We develop a framework for the analysis of strategic interactions under the constructive preference perspective à la Kahneman and Tversky formalized in the Type Indeterminacy model. The players are modeled as systems subject to measurements and characterized by quantum-like uncertain preferences. The decision nodes are modeled as, possibly non-commuting, operators that measure preferences modulo strategic reasoning. We define a Hilbert space of types spanned by the players' eigentypes representing their potential preferences in different situations. We focus on pure strategy TI games of maximal information where all uncertainty stems from the intrinsic indeterminacy of preferences. We show that preferences evolve in a non-deterministic manner with actions along the play: they are endogenous to the interaction. We propose the notion of cashing-on-the-go to compute a player's utility, and the Type Indeterminate Nash Equilibrium as a solution concept relying on best-replies at the level of the eigentypes. We illustrate an example exhibiting the phenomenon of the manipulation of rivals' preferences.

Keywords: Type Indeterminacy; Superposition of preferences; Actions as projections; Hilbert space of types; Cashing-on-the-go; Type Indeterminate Nash Equilibrium.

JEL classification: C79,C72, D03, D89.

> *Theories are nets cast to catch what we call 'the world':*
> *to rationalize, to explain, and to master it.*
> *We endeavour to make the mesh ever finer and finer.*
>
> Karl Popper, in The Logic of Scientific Discovery (1935).

1. Introduction

This paper belongs to a recent and rapidly growing literature where formal tools of Quantum Mechanics are proposed to explain a variety of behavioral anomalies in social sciences and in psychology. After the fundamental paper by Deutsch (1999), Busemeyer et al. (2006) present a quantum dynamical model for decision-making processes. Busemeyer and Wang (2007) and Busemeyer et al. (2008) show how processing information in a quantum fashion can overcome limitations of Markov models. Franco (2009) explains the conjunction fallacy in terms of the quantum interference effect. Danilov and Lambert-Mogiliansky (2008, 2010) define the basic properties of non-classical systems applied to social sciences as a formulation for bounded rationality, and extend Savage's framework for decision-making under uncertainty to the non-classical domain. Lambert-Mogiliansky et al. (2009) show how the mathematical formalism of quantum mechanics can be fruitful in providing explanation to violations of transitivity in decision-making, and of the Principle of Indifference of Irrelevant Alternatives.[1]

To many people it may appear unmotivated or artificial to turn to Quantum mechanics when investigating human behavioral phenomena. However, the founders of Quantum Mechanics, including Bohr (1991) and Heisenberg (2000), early recognized the similarities between the two fields. In particular Bohr was influenced by the psychology and philosophy of knowledge of Harald Höffding.

The similarity stems from the fact that in both fields *the object of investigation cannot (always) be separated from the process of investigation.*[2]. Quantum Mechanics and in particular its mathematical formalism was developed to respond to that epistemological challenge (see the Introduction in Bitbol (2009) for an enlightening presentation).

The use of quantum formalism in game theory was initiated by Eisert et al. (1999), who study how the extension of classical moves to quantum ones can affect the analysis of a game.[3] Another example is La Mura (2005) who investigates correlated equilibria with quantum signals in classical games. Whether and when the use of quantum strategies (or strategies using quantum signals) can bring something truly novel to Game Theory has been discussed by Levine (2005), and Brandenburger (2010).

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[1] See the books by Busemeyer and Bruza (2012), Khrennikov (2010), and Haven and Khrennikov (2012) for monographic overviews of other achievements in quantum-like models for decision-making problems.

[2] In words of Bohr (1950, p. 52), "the impossibility of separating a behavior of atomic objects from the interaction of these objects with the measuring instruments which serve to specify the conditions under which the phenomena appears" represents an epistemological challenge shared with other disciplines. In psychology, investigating a person's emotional state affects the state of the person. In social sciences, *revealing* one's preferences in a choice can affect those preferences: "There is a growing body of evidence that supports an alternative conception according to which preferences are often constructed –not merely revealed– in the elicitation process. These constructions are contingent on the framing of the problem, the method of elicitation, and the context of the choice" (Kahneman and Tversky, 2000, p.525)

[3] From a game-theoretical point of view the approach consists in changing the structure of the strategy space, and thus the interest of the results lies in the appeal of these changes.

Departing from the so-called quantum game approach, our approach is based on the idea that players' preferences (types), rather than the strategies they can choose, can feature non-classical (quantum) properties. This idea is formalized in the Type Indeterminacy (TI) model of decision-making introduced by Lambert-Mogiliansky et al. (2009), and Kvam et al. (2014) provide empirical evidence supporting the TI model.

The concept of TI agents has been proposed as a theoretical framework for modelling the Kahneman-Tversky man, *i.e.*, the *constructive preference perspective*. The novelty of our contribution is the proposal of encoding the preference relations over the set of possible alternatives in a quantum-like fashion, *i.e.*, orthonormal basis and Hermitian operators; and not the game strategies as was done in previous studies.

This paper defines the basic elements of a theory of games with Type Indeterminate players. We aim at a self-contained theoretical contribution to the fields of Behavioral Economics and Game Theory proposing a tractable way of formalizing contextual types. We keep the notation as close as possible to the standard one in Game Theory and we consider only basic knowledge of vector spaces to be required for the reading. The rest of the paper is structured as follows.

In Section 2, we first establish how a finite number of mutually exclusive preference relations of a player (called eigentypes) in a given decision-situation can be modeled as an orthogonal basis of a vector space: the Hilbert Space of Types. We introduce two postulates: (*i*) the superposition of preferences is a valid type for a player, and (*ii*) how the choice of actions affects these preferences. They are inspired by the Principle of Superposition and the Postulate of Measurement and Observables in Quantum Mechanics, respectively. We discuss the probabilistic interpretation of this framework and how the intrinsic uncertainty in preferences relates to a model of non-commuting decision-situations.[4]

Section 3 introduces the basic elements for the study of TI games, the interactions between TI players. The preferences of a TI player are represented as a unit length vector giving her type as a linear combination of the elements of an orthonormal basis. When the player faces several non-commuting decision-situations, the Hilbert Space of Types admits several orthonormal bases of eigentypes related to each other by means of basis transformation matrices. These are basic operations in any linear space. In this contribution, we focus only on TI games with maximal information, so that the whole structure of eigentypes and their relations is known.

The timing of a TI game defines a partially ordered set of decision-situations which may imply a sequence of projections of the preferences along the path defined by the choice of actions. This 'evolution' of the preferences is endogenous to the interaction between TI players and is, in general, non-deterministic.

We discuss in Section 4 a definition for the Type Indeterminate Nash Equilibrium (henceforth TINE). The TINE is proposed as a solution concept for TI games whose main feature is that the choices of actions are best-responses at the level of the eigentypes, while the overall utility for the player is computed at the level of the type. Generally, the type is a linear superposition of several eigentypes, and the composition of this combination is affected by the choice of actions. The idea of *cashing-on-the-go* is formalized in Assumptions

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[4]Under some simplifying conditions, a TI model can yield the same predictions as a 'classical' Bayesian framework.

1 and 2. A TINE specifies the equilibrium strategy of each player as the collection of the best-replies of all possible eigentypes of each player, as well as the expected resulting profile of preferences of the players as a consequence of their plays along the game. A TINE implies an expected utility level for all the players.

Section 5 provides an illustrative example. We consider a simple game with two players: Alice and Bob. On the one hand, Alice behaves as a 'standard' player with stable preferences since she faces only commuting decision-situations. On the other hand, Bob exhibits the richness of a TI player. We show how Alice can determine the path of measurements of Bob's preferences with respect to two non-commuting decision-situations because she has the opportunity to make the first move.

This example illustrates how to operate with the computational machinery in a Hilbert Space of Types, and it emphasizes the notion of actions as projectors in the space of preferences. We show how, in the TINE, Alice departs from the classical equilibrium play because of the strategic value of manipulating Bob's intrinsically indeterminate preferences. We conclude in Section 6 with some final remarks and the discussion of several aspects for further study.

2. The Hilbert Space of Types

In this paper an agent is characterized by her type. The type captures maximal information about the agent's preferences over the actions in interactive decision-situations. The Type Indeterminacy approach defines an agent as a measurable system. Such a system is characterized by its state: an element in a Hilbert space endowed with a set of operators (measurements).[5]

2.1. A Type as a Superposition of Preferences:

Let O be the set of possible outcomes of an interaction between players. Then, a *preference relation* of player i over the set of outcomes O is a binary relation denoted by $\succsim_i (O \times O)$.[6] With $\succsim_i$ being a preference relation, a utility function u_i representing $\succsim_i$ is a real-valued function defined over O and satisfying $u_i(o_1) \geq u_i(o_2) \Leftrightarrow o_1 \succsim_i o_2 \ \forall o_1, o_2 \in O$.

Definition 2.1 (Orthonormal Basis of Preferences). Let $R_i(O) = \{\succsim_i^{(n)}\}_{n=1}^N$ be a set of N different preference relations over a set of outcomes O. We associate them in a bijection with a set $\Theta_i = \{\theta_i^{(n)}\}_{n=1}^N$ of orthonormal vectors in a Hilbert space.[7] Then, Θ_i is an orthonormal basis of a Hilbert space of dimension N.[8]

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[5]For a more general discussion in terms of ortholattices, see Danilov and Lambert-Mogiliansky (2008). We restrict ourselves to separable and finite-dimensional Hilbert spaces defined over the field of real numbers.

[6]Given two elements $o_j, o_k \in O$, $o_j \sim o_k$ denotes the indifference between both of them defined as $o_j \succsim_i o_k$ and $o_k \succsim_i o_j$, and $o_j \succ_i o_k$ denotes the strict preference for o_j defined as $o_j \succsim_i o_k$ and not $o_k \succsim_i o_j$.

[7]Orthonormality for the set of vectors requires $\langle \theta_n, \theta_m \rangle = \delta_{nm}$, where δ_{nm} is the Kroenecker symbol.

[8]As a simplifying remark, a preference relation defined over a set of M alternatives is, generally, composed of a list of the order of $C_{M,2}$ elements, the combinatorial number counting how many different pairs we can form with the collection of M elements. Imagine a decision-maker i who can conceive only two monotonic ways of ranking the elements: either in a 'positive' way $M \succsim_i^{(+)} \dots \succsim_i^{(+)} 1$ or in a 'negative' way $1 \succsim_i^{(-)} \dots \succsim_i^{(-)} M$. Then, the number of eigentypes required to describe our thought-agent is just $N = 2$, because there are

From the bijection $R_i(O) \leftrightarrow \Theta_i$, we equivalently refer to the preference relations of a player and to the corresponding vectors of the orthonormal basis. We denote both of them by Θ_i from now on to lighten the notation.

Definition 2.2 (Hilbert Space of Types). Let Θ_i be the set of orthonormal vectors in a bijection with the set of preference relations of player i. Then, we denote by $\mathcal{T}_i$ the N-dimensional Hilbert space of the types of player i, $\mathcal{T}_i \equiv \mathrm{Span}(\Theta_i)$, spanned by the orthonormal basis of preference relations of player i.

Let a player i be described in terms of a Hilbert space of types $\mathcal{T}_i$. Then, the type of the player is fully characterized by a state-vector $t_i \in \mathcal{T}_i$ such that $\|t_i\| = 1$. A proper representation of the state of a system is expressed in terms of vectors of the Hilbert space with unit length as a requirement to be a probability framework. All the elements of the form λt_i (with $\lambda \neq 0$) represent the same state of the system as t_i, with $\|t_i\| = 1$.

Postulate 1 (Superposition of Preferences). Let $\mathcal{T}_i$ be the Hilbert space of types spanned by the possible preferences of the player, $\{\theta_i^{(n)}\}_{n=1}^N$. Then, every type of the form

$$t_i = \sum_{n=1}^N c_n \theta_i^{(n)}, \text{ with } c_n \in \mathbb{R}, \text{ and } \sum_{n=1}^N c_n^2 = 1, \tag{1}$$

is also a unit length vector belonging to the same Hilbert space, $t_i \in \mathcal{T}_i$. Hence, any linear combination of preferences (eigentypes) of a player is itself a proper type of the player.

Postulate 1 represents a major departure from the standard models in Decision and Game Theory, where the linear combinations of types are understood as tools for computing expected values when the other players lack information about the proper type of a player. In a TI game, any superposition is a proper type itself, meaning that the type of an agent is generally characterized by several mutually exclusive preference relations.[9] For the rest of this paper we use the term *type* of a player to denote *a superposition of mutually exclusive preference relations*, which are referred to as the *eigenpreferences* or the *eigentypes* associated to each decision-making node.

2.2. Strategic Reasoning, Actions and Projections:

The players' choices of actions reflect the type of the players. Nevertheless, when observing the choice of an action, we do not directly learn about the preferences: we observe the type *modulo strategic reasoning*. Note that this is standard in game theory but it acquires a new meaning in our context. The notion of intrinsic uncertainty entails that the players' type must be defined together with the decision-nodes to which they are confronted along the game.[10]

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only two possible preference relations even though each of this preference relations contains many pairwise comparisons between the elements in the outcome set.

[9]At this initial stage of our research, we restrict the model to pure types. We do not consider mixed types (represented by density matrices) that are expression of incomplete information. We deal exclusively with situations of *maximal information*, where all the uncertainty is intrinsic.

[10]We return to the distinction between the observed choice and the underlying preferences in more detail later.

For any decision-situation d, player i has a set of possible actions $A_i(d)$ among which she can choose. Taking actions reveals information about the underlying preferences which are present in the type of the player. It is therefore natural to understand the actions as related to the outcomes of some sort of measurements of the type. Let the set $\Theta_i(d)$ contain those preference relations (eigentypes) that can be actualized for player i in the state of preferences (type) t_i when facing a decision-situation d. Then, the choice of a particular action $a_i(d)$ implies the transition of player i's type from the initial state t_i to an outcoming state t_i', resulting from the measurement process.

We model this transition as a projection onto the subspace of types spanned by the eigentypes supporting the chosen action,[11] as a consequence of strategic reasoning. Formally,

Definition 2.3 (Action as Projector). Let $a_i(d) \in A_i(d)$ be a particular action that can be chosen by player i at a decision-situation d, and let us assume that $a_i(d)$ is the preferred action for a certain number $M \leq N$ of eigentypes $\{t_1, \ldots, t_M\} \in \Theta_i(d)$. Then, the matrix

$$P_{a_i(d)} = \sum_{t_1,\ldots,t_M} t_m t_m^T \tag{2}$$

is the projector associated to the action $a_i(d)$.[12]

Postulate 2 (Actions affect Preferences). For player i facing a given decision-situation d, the chosen action $a_i(d) \in A_i(d)$ is the outcome of a measurement of her preferences. The type of player i after making her decision in decision-situation d is:

$$t_i' = \frac{P_{a_i(d)} t_i}{\|P_{a_i(d)} t_i\|}, \tag{3}$$

where t_i is the type before making the decision.

If one action is preferred only by one particular eigentype, the outcoming preferences of the player only contain information of the eigentype that was actualized with the selected action. Usually, an action may be preferred by *several* of the eigentypes belonging to the initial superposion. The outcoming preferences of the player are then a superposition of all those eigentypes. If one and the same action is preferred by all the eigentypes, the projector is the identity matrix and the outcoming preferences of player i after taking such action, t_i', are identical to the incoming preferences, t_i.

The projections corresponding to the possible actions are orthogonal, and the Hilbert space of types $\mathcal{T}_i$ can be decomposed as the direct sum of the subspaces associated to each of the actions (this relates to spectral decomposition). We note that in case we were interested in a single move corresponding to a single measurement, Postulate 2 would imply Bayesian updating.

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[11] This follows in the spirit of Lüders' postulate, a generalization of the so-called von Neumann's Postulate for Pure States, which applies only for nondegenerate spectrums of observables. We consider our manuscript presents a self-contained exposition of the use of the Hilbert spaces for describing the space of types in Game Theory. Nevertheless, for the reader deeply interested in the interpretation of the *measurement postulate* in Quantum Mechanics, we suggest the discussion in *Elementary Analysis of the Measurement Process*, Chapter 8 of *The Logic of Quantum Mechanics* by E.G. Beltrametti and G. Cassinelli (1981), as well as in A. Khrennikov, *von Neumann and Lüders' Postulates and Quantum Information Theory.* International Journal of Quantum Information, 7, pp.1303-1311 (2009).

[12] The row-vector t_m^T is the transposition of $t_m \in \mathcal{T}_i$, and $P_{a_i(d)}$ is a $N \times N$ matrix.

2.3. Probabilistic Content of a Type t_i:

Let the state of preferences of player i be given by the type t_i as a superposition of the eigentypes in $\Theta_i(d)$, according to Postulate 1. Hence, strategic reasoning performed at the level of the eigentypes determines which action each eigentype chooses if given a chance to act. The actual observed action chosen by player i depends on the probability of actualization of each eigentype, defined by her incoming type t_i expressed in terms of the eigenvectors corresponding to the actions available in decision-situation d.

Definition 2.4 (Probability of Actualization)**.** Let t_i be the preferences vector-state of player i when facing a decision-situation d expressed as in Postulate 1, $t_i = \sum_{n=1}^{N} c_n \theta_i^{(n)}$, with $c_n \in \mathbb{R}$, and $\sum_{n=1}^{N} c_n^2 = 1$; and let each of the possible actions $a_i(d) \in A_i(d)$ be associated to an orthogonal projector $P_{a_i(d)}$ reflecting the strategic reasoning. Then, a particular action $a_i(d)$ will be selected by player i in decision-situation d with a probability given by

$$p\big(a_i(d)\big|t_i\big) = \|P_{a_i(d)} t_i\| = \sum_{m=1}^{M} c_m^2 \leq 1, \tag{4}$$

with $\{c_m\}_{m=1}^{M}$ being the linear coefficients of those eigentypes $\{\theta_i^{(m)}\}_{m=1}^{M} \subseteq \Theta_i(d)$ for which the action $a_i(d)$ is the preferred one.

2.4. Intrinsic Uncertainty of Preferences:

The Hilbert space of types provides a suitable framework for describing agents with intrinsic uncertainty involved in the strategic interactions. We have to emphasize in this section that by *intrinsic uncertainty* we mean a kind of uncertainty different from the uncertainty due to incomplete information and different from the strategic uncertainty which arises when considering mixed strategies (we deal only with pure strategies in this paper).

The realization of a particular action $a_i(d)$ when player i is making a decision in decision-situation d will, in general, alter the preferences of the agent with respect to another decision-situation d', as a consequence of projecting the preferences t_i of the player onto the subspace of types supporting $a_i(d)$.

Let $\Theta_i(d)$ and $\Theta_i(d')$ be two sets of eigentypes representing the preference relations for player i facing two different decision-situations d and d', respectively. Let us assume for clarity of the exposition that both sets contain the same number N of elements.[13] In the particular case that the realized action $a_i(d)$ is supported by only one eigentype θ_i, one may think that this measurement process clears *all* the uncertainty about the preferences of the agent, but this is not true when the decision-situations are incompatible or, equivalently, they represent non-commuting measurements of the agent's type (in plain words, if order effects may arise).

The vectors $\theta_i' \in \Theta_i(d')$ then form an alternative orthonormal basis spanning $\mathcal{T}_i$. The two sets of vectors $\Theta_i(d)$ and $\Theta_i(d')$ are related by a basis transformation matrix, $B_i^{(d',d)} : \mathcal{T}_i \to \mathcal{T}_i$, such that every vector represented in terms of coordinates with respect to one basis is uniquely determined in terms of the other one. When the chosen action $a_i(d)$ is supported by only

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[13]We shall deal with the problem of coarse measurement in the future.

one eigentype, let us say $\theta_i^{(m)}(d)$, preferences of player i are fully determined with respect to decision-situation d. But player i's preferences when facing the next decision-situation d' are given by $t_i' = B_i^{(d',d)}\theta_i^{(m)}$.

Thus, the vector t_i' is, in general, a superposition of several eigentypes in $\Theta_i(d')$. When the decision-situations d and d' commute, the dimensionality of the type space is given by the tensor product of the type space corresponding to each decision-situation. If this is the case, the corresponding preferences are characterized by some independence and the analysis can be fully classical.[14]

3. Basic Model for Games with Quantum-like Players

3.1. Games with Type Indeterminate Players

A TI game is an interacting situation where the agents are modeled as Type Indeterminate players. A TI player i is a decision-maker facing a set of decision-situations D_i, for which player i's preferences are actualized when the decision is made, interpreted as the outcome of a measurement process, following Postulate 2. The following elements define a TI player:

1. The set D_i of decision-situations in which player i has to take an action.
2. The collection $\Theta_i = \{\Theta_i(d)\}_{d \in D_i}$ of the sets $\Theta_i(d)$ of eigentypes giving the orthonormal basis of eigenpreferences of player i in each decision-situation $d \in D_i$, as in Definition 2.1.
3. The initial type of player i, $t_i^0 \in \mathcal{T}_i$, given as an element of the Hilbert space of types spanned by the basis of the decision-situations corresponding to player i, Θ_i, as in Definition 2.2.

A TI game is of maximal information if:

1. Every decision-situation of the game is identified and common knowledge.
2. Every orthonormal basis $\Theta_i(d) \in \Theta_i$ associated to each decision-situation $d \in D_i$ is common knowledge, as well as the relation among them, $B_i^{(d';d)}$, for every pair $d, d' \in D_i$.
3. The initial type t_i^0 of every player is known and common knowledge.

3.2. Basic model of a TI Game of Maximal Information:

A TI game is defined as $\Gamma \equiv \left\langle I, \mathcal{D}, \mathcal{T}, \left\langle t_i^0, A_i, \{u_i(\cdot)\}_{d \in D_i} \right\rangle_{i \in I} \right\rangle$, where:

1. I is the set of TI players taking part in the game, each of them labeled by $i \in I$.
2. $\mathcal{D} = \mathcal{D}(D, \leq)$ is the partially ordered set of the decision-situations in the game. This ordering comes from the time structure of the game, which equips the set $D = \cup_{i \in I} D_i$ with an ordering relation $\leq$ of temporal nature, and where D_i is the set of decision-situations in which player i has to participate.
3. $\mathcal{T} \equiv \bigotimes_{i \in I} \mathcal{T}_i$ is the Hilbert space of types describing the type for all the agents in the game, with the set $\Theta = \{\Theta_i\}_{i \in I}$ of all the orthonormal basis associated to each player's eigenpreferences.

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[14]Note that commuting decision situations do not preclude statistical correlations.

4. t_i^0 gives the initial type of each player i.

5. $A_i = \{A_i(d)\}_{d \in D_i}$ is the collection of sets specifying the available actions for each player i, at each of her decision-situations.

6. $u_i(\cdot) : \Theta_i(d) \times A_i(d) \times A_{-i}(d) \to \mathbb{R}$ is the payoff function for each player i.[15]

3.3. Paths of Projection:

The already defined partially ordered set $\mathcal{D}$ is obtained when considering the collection of decision-situations of the game together with the ordering imposed by the timing of the game. After understanding the actions as projections in the space of preferences (see Postulate 2), we find a richer structure that can be defined over $\mathcal{D}$. Given the specification of payoffs $\{u_i(\cdot)\}_{d \in D_i}$, we have interpreted the actions as the projections defined by the underlying structure of preferences of the agents (see Definition 2.3), modulo strategic reasoning. Then, we can associate each decision-situation node $d \in D$ to a projector acting over the type of the players who make their decision at the node d.

It follows directly that the partially ordered set of decision-situations $\mathcal{D}$, together with the specification of the preferred action for each of the available eigentypes, defines a partially ordered set of projectors. In order not to overload the notation, we assume from now on that $\mathcal{D}$ can refer also to the partially ordered set of projectors actualized along the path of the game.

Therefore, for every TI game Γ as previously defined, we build the set Σ of all possible paths of measurement along the game, from the root to each of the possible end nodes of the game. For a given path $\sigma_l \in \Sigma$, of length L, we have a chain of projectors of length up to L associated to the path along the game $\sigma_l \mapsto (P_{a_i(d_1)}, P_{a_i(d_2)}, \ldots, P_{a_i(d_L)})$, with the index $d_1 < d_2 < \ldots < d_L$ giving the order in which the L decision-situations are reached when the chain describes the play of the game.

3.4. On the *Evolution* of TI Preferences along the Game:

In standard Game Theory the preferences of a given agent are expressed in terms of a *utility function* defined only over the different end nodes of the game. The purpose of this section is to clarify certain aspects characterizing the theory of TI games which departs from both the standard and the non-deterministic approaches to the definition of the utility which clearly rely on the temporal invariance of the players' preferences.

A main feature of the TI games is that the preferences are *endogenous*, they are a part of the outcome of the game as a result of the choices that are made. We can say that the preferences of the agents are initially the motivation for but finally the consequence of the choices along the path of decisions. The chain of projections defined in the play of the game determines the *evolution* of the players' preferences. Within this framework, the possibility for a *strategic manipulation* of the other players' preferences arises as a new field of interaction among the agents, as we will see in the example below.

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[15] Just for notation, $A_{-i}(d)$ represents the actions available for all the players other than i that are involved in the decision-situation d. Then, $u_i[\theta_i(d), a_i(d), a_{-i}(d)]$ gives the utility associated to eigenpreferences $\theta_i(d)$ when facing an interaction (a_i, a_{-i}) in the decision-situation d.

The idea of preferences that evolve along with the taken action is not completely new. It was initially discussed from the point of view of consistent planning and welfare economics. See, *e.g.*, the discussions by Schoeffler (1952) and Harsanyi (1953). Peleg and Yaari (1973) defined the notion of *equilibrium consumption plan* for the optimal behavior of agents immersed in multi-stage decision processes with certain preferences that evolve deterministically.

This original approach referred to as 'agent-form games' consists in defining a sequence of decision-makers, one for each decision-stage. A thorough discussion on credible equilibria in agent-form games is given by Ferreira et al. (1995). This seminal discussion suggests considering *situations where a player is a group of individuals. [...] Such a player is to some extent a decision-making unit, but it does not have a utility function of its own* (Ferreira et al., 1995).

At first glance, one might think that a TI game could be properly described also as an *agent-form* game as in Ferreira et al. (1995) but, as we shall attempt to show below, a proper TI game has a richer structure because it allows for the strategic manipulation of preferences. Moreover, because preferences are represented in a Hilbert space of types, superposition of *a priori* incompatible aspects (eigentypes) of the personality of the agent are by definition valid types and they may interfere when computing the overall utility level of the agent who is facing the different decision-situations.

4. The Type-Indeterminate Nash Equilibrium

We have defined how the preferences of the TI players evolve on the path of the game along with the actions that are taken. Now, we propose a solution concept which is an extension of the classical notion of Nash Equilibrium to games with TI players. The strategies are defined at the level of the potential eigentypes and that is also where the equilibrium reasoning takes place. The definition of utility integrates the consequences for the whole individual – beyond its current potential incarnation (eigentype) – for the rest of the continuation game, including how the choice of an action conditions the future preferences. For this, let us begin with the novel concept of *cashing-on-the-go*.

4.1. *Cashing-on-the-go*:

In a Hilbert space of types, the superposition of eigentypes is itself a proper type of the agent (Postulate 1). If the action taken by a player is the best-reply of several eigentypes, the resulting superposition must be the basis for computing the utility level of the agent. Since taking actions generally alters the type of the agent, we propose that in order to compute the utility value of an action one should not wait until the last node of the game. This means that we consider that the agent acts as if he was cashing in utility along with the game. He may actually experience utility after taking the action. But we can also think that he 'savors' (experiences before the actual outcome).

There are several reasons for this. Most importantly, since the strategic reasoning is performed by the eigentypes (who know their preferences as in a standard game) they have to compute the utility associated with the different actions in order to identify the best-reply. If they reasoned from the end node, their own preferences would play no role and

the computation would be extremely complex as they would have to calculate at each step the expected type of the agent arising from the possible actions to determine the end node (expected) type. Moreover, such an approach would not allow for the central novel features brought about by Type Indeterminacy.

To understand better, let us consider a path $\sigma_l \in \Sigma$ along a TI game Γ, containing two particular decision-situations d and d' such that player i has to take an action for d some steps before she takes an action for d', following the timing of the game represented by the ordering $\leq$. According to Postulate 2, the state-vector t_i giving player i's preferences is affected by the course of actions such that $t_i^{(\text{before } d)} \neq t_i^{(\text{after } d)} \neq t_i^{(\text{before } d')} \neq t_i^{(\text{after } d')}$ holds.[16]

Thus, we propose to consider the idea of *cashing-on-the-go*, so that the payoffs, rewards or punishments are received when the preferences are actualized through the actions which are taken. Therefore, for each possible play of the game given as a path $\sigma_l \in \Sigma$, we can define the total utility of the path as the collection of all the contributions of the payoffs received when solving each decision-situation with respect to the preferences governing the affinities of the player in each step.

In plain words, we model the reader of this paper enjoying the reading at present time, as well as enjoying past readings in the past (now you only enjoy the memory of them) and future readings in the future. This is opposed to the usual convention with all enjoyment being at the end of the game of an agent's life when final preferences of the player, in general, do not coincide with the preferences motivating each particular choice along the path of life. Experience conditions and modifies preferences, and the TI framework mirrors and formalizes this effect.

4.2. Solution Concept in a TI Game of Maximal Information:

The optimal play in a TI game arises from the strategic reasoning at the level of the eigentypes. This implies the strategies are defined such that one action is selected by each eigentype of every player.[17] We propose the notion of Type Indeterminate Nash Equilibrium (TINE) as a solution concept for the TI games. A TINE builds on the notion of the players' mutual best-replies as the standard Nash equilibrium does. The major distinction now is that the best-replies are computed at the level of the eigentypes of each player.[18] The actual play of a player is, in addition, determined by the probability for the different eigentypes to actualize according to Definition 2.4.

Following Postulate 2, the actions taken in the decision-situations along the play of the game define a path of projection, as discussed in Section 3.3. This path of projection determines the state of the outcoming preferences of the players after the interactions. Resulting preferences of a TI player are endogenous to the game, they arise in the process of interaction. The outcome of a TI game includes the payoffs of the players, and the outcoming profile of preferences.

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[16]The particular exception $t_i^{(\text{after } d)} = t_i^{(\text{before } d')}$ is found if d strictly precedes d' in the path, or if all of the intermediate decision-situations between d and d' are trivially associated to the identity projector.

[17]Recall that for this first work we consider only pure strategies.

[18]The situations that can be analyzed with the notion of Nash Equilibrium are contained in the TI framework as an oversimplified case where only one eigentype exists for each player and therefore, the type and the eigentype trivially coincide.

We proceed in this section as follows:

1. First, we define a TINE as the profile of the (pure strategy) best-reply of all the eigentypes of all players, a complete algorithm for the players' action in the game. See Definitions 4.1 to 4.3.
2. Second, we compute the overall utility of the paths in Definitions 4.4 and 4.5. We propose the *cashing-on-the-go* for solving TI games in the Assumption 2.[19]
3. Third, we compute the profile of outcoming types of the players which, in general, are different from the initial types. The play alters the preferences in accordance with Postulate 2 (Projection) as discussed in Section 3.3. This is formalized in Definition 4.6.

I. Definition of TINE:

In Game Theory, the definition of a *strategy* implies an algorithm specifying what actions are chosen by a player i in every possible situation in the game. An action has to be assigned to every node, irregardless of whether some previous action forbids some nodes to be reached: the strategy specifies a fully fledged and contingent instruction for the complete play. When considering TI players, a strategy s_i in a TI game has to explicitly specify the action that is selected by every possible eigentype that a player i can incarnate in each decision-situation.

Definition 4.1 (Strategies of a TI Player). Let a TI player i face a number K of decision-situations with a set $\Theta_i(d_k)$ of eigentypes associated for each decision-situation $d_k \in D_i$. A strategy of a TI player s_i is a complete algorithm for her play in the TI game, and it contains one action $a_i\big(\theta_i^{(n)}(d_k)\big)$ for each and every of the N eigentypes $\theta_i^{(n)} \in \Theta_i(d_k)$ associated to each of the K decision-situations $d_k \in D_i$ in which player i takes part in the game. Therefore, s_i is a collection of $N \times K$ elements,

$$
\begin{aligned}
s_i \;=\; & \Big(a_i\big(\theta_i^{(1)}(d_1)\big),\, a_i\big(\theta_i^{(2)}(d_1)\big),\, \ldots,\, a_i\big(\theta_i^{(N)}(d_1)\big);\, \ldots \\
& \underbrace{a_i\big(\theta_i^{(1)}(d_k)\big),\, a_i\big(\theta_i^{(2)}(d_k)\big),\, \ldots,\, a_i\big(\theta_i^{(N)}(d_k)\big);}_{\text{One action for each of the } N \text{ eigentypes in each } d_k}\, \ldots \\
& a_i\big(\theta_i^{(1)}(d_K)\big),\, a_i\big(\theta_i^{(2)}(d_K)\big),\, \ldots,\, a_i\big(\theta_i^{(N)}(d_K)\big)\Big).
\end{aligned}
$$

(5)

Assumption 1. The optimal strategy of a TI player arises from the optimality of the actions at the level of the eigentypes: every eigentype of every player best-responds to the expected play of the other players computed from the best-replies of the eigentypes that enter their current type, weighted by the coefficients of superposition in every decision-situation (node) of the game.

Definition 4.2 (Best-reply of an Eigentype). Let $A_i(d)$ be the set of available actions for player i facing decision-situation $d \in D_i$, and let $A_{-i}(d)$ be the set of available actions for

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[19]*Cashing-on-the-go* has been introduced to deal with the fact that the initial preferences of TI players are, in general, altered along the path of the game as discussed in Section 4.1.

all the other players $I \setminus \{i\}$ taking part in the interaction. For every given action profile of the other players $a_{-i}(d)$, we can define the preferred action (best-reply) of each one of the eigentypes $\theta_i^{(n)} \in \Theta_i(d)$ of player i in the decision-situation d. We denote this by $a_i^*\big[\theta_i^{(n)}, a_{-i}\big]$ such that

$$a_i^*\big(\theta_i^{(n)}, a_{-i}\big) \in \arg \max_{a_i(d) \in A_i(d)} u_i\big[\theta_i^{(n)}; a_i(d), a_{-i}(d)\big]. \tag{6}$$

Definition 4.3 (TINE). A particular profile of strategies $(s_i^*; s_{-i}^*)$ constitutes a Type Indeterminate Nash Equilibrium of a TI game when they are the collection of all the eigentypes of every player best-responding to every eigentype of the other players, in the sense of Definition 4.2, so that

$$u_i\big[\theta_i^{(n)}; a_i^*(d), a_{-i}^*(d)\big] \geq u_i\big[\theta_i^{(n)}; a_i(d), a_{-i}^*(d)\big] \; \forall \, a_i(d) \in A_i(d) \tag{7}$$

holds for every eigentype $\theta_i^{(n)} \in \Theta_i(d)$ of every player $i \in I$, at every decision-situation $d \in D_i$.

We can denote by $\sigma^* = \sigma(s_i^*; s_{-i}^*)$ the equilibrium path of the game obtained from the combination of these strategies, optimized at the level of the eigenpreferences. The outcome of the TINE includes: (i) the overall utility level of every TI player, and (ii) the final state of preferences of every TI player (see Definitions 4.5 and 4.6 below).

II. Utility of the Outcome:

According to Postulate 2, when the preferred action $a_i^*(d)$ is taken by player i, her outcoming preferences t_i' are the result of projecting the incoming type t_i onto the M-dimensional subspace ($M \leq N$) of the preference relations (eigentypes) supporting that particular action as their preferred one in the strategic interaction. And according to Postulate 1, this superposition of eigenpreferences is now also a proper type of the player given by

$$t_i' = \sum_{m=1}^{M} c_m' \theta_i^{(m)}, \; \text{with } c_m' \in \mathbb{R}, \; \text{and } \sum_{m=1}^{M} c_m'^2 = 1. \tag{8}$$

Definition 4.4 (*Cashed* Utility of a Preferred Action). Let player i face a decision-situation d with the preferences given by a type $t_i \in \mathcal{T}_i$, and let $a_{-i}^*(d)$ be the actions chosen by the other players. Then, the utility *cashed* by player i when taking the action $a_i^*(d)$ as in Definition 4.2 is the weighted utility of the M eigenpreferences where the weights are given by the coefficients in the superposition:

$$u_i\big[t_i'; a_i^*(d), a_{-i}^*(d)\big] = \sum_{m=1}^{M} c_m'^2 \, u_i\big[\theta_i^{(m)}; a_i^*(d), a_{-i}^*(d)\big]. \tag{9}$$

Note that the utility defined in the equation (4.4) is linear in the probabilistic content of the outcoming preferences (compare $c_m'^2$ to Definition 2.4).

Assumption 2. We compute the overall utility of the players involved in a TI game as the addition of the *local* utilities cashed by the type of each of the players in the decision-situations along the path of play.

Definition 4.5 (Total Utility of a Path)**.** Let $\sigma_l \in \Sigma$ be a path $\sigma_l = \{d_1, \ldots, d_L\}$, induced by the strategy profile $(s_i^*; s_{-i}^*)$. Then, the total utility for player i in the play following path σ_l is given by

$$u_i[\sigma_l] = \sum_{d_l \in \sigma_l} u_i\big[a_i^*(d_l), a_{-i}^*(d_l)\big], \tag{10}$$

with the utility of the preferred actions given in Definition 4.4.

When some paths become 'branches' including moves of chance, the utility shall be considered in expected terms.

III. Outcoming Preferences:

Definition 4.6 (Outcoming State of Preferences)**.** Given a path of play σ_l, and the state-vector of initial preferences of a TI player i denoted by $t_i^0 \in \mathcal{T}_i$, the outcoming preferences of the TI player i are computed by the consecutive application of the projections as a consequence of the actions taken along the path of play:

$$t_i^{(\mathrm{end})} = \prod_{d_l \in \sigma_l} P_{a_i^*(d_l)} t_i^0 \ \ \text{for every player } i \in I. \tag{11}$$

After the dicussion presented up to this point of the paper, equation (11) translates into computations by means of Definition 2.3 and Postulate 2. The notation $\prod_{d_l \in \sigma_l} P_{a_i^*(d_l)} t_i^0$ for the subsequent application of projectors has to be understood in the sense of the composition of operators, $P_{a_i^*(d_L)} \cdots P_{a_i^*(d_2)} P_{a_i^*(d_1)} t_i i^0$. This determines a process of all the players *updating their knowledge* about the state of preferences of themselves and the other players in the game. The projections reflect the evolution of the types resulting from the choice of actions (modeled as the measurement of preferences) originated by the eigentypes' strategic reasoning, so the rule for updating knowledge is a constituent part of the solution of the game.

5. Illustrative Example

The following example is motivated by the one presented in Lambert-Mogiliansky (2010). *The story:* Alice (player A) is a Tenured Professor at University who starts working with her new student Bob (player B). Bob is finishing his Master studies and later on he can choose to work with Alice for a PhD. For the long-term, Alice wants Bob to agree to cooperate with her in a very specific idea of research. Bob is indeterminate with respect to his willingness to engage in this specific topic which appeals to open-mindness. Bob is also indeterminate with respect to his taste for personal challenges.[20]

First: Alice is currently Bob's Master thesis advisor and she has two different ideas to propose him for his Master thesis: either a standard problem (S) or an intricate one (I).

Second: once Alice offers a specific topic, Bob can work on his Master thesis either on a routine basis (R) or adopting a more creative approach (C). Bob's preferences regarding his

---

[20]We assume the open-mindness and the taste for personal challenge are not the same psychological features but they are somehow related.
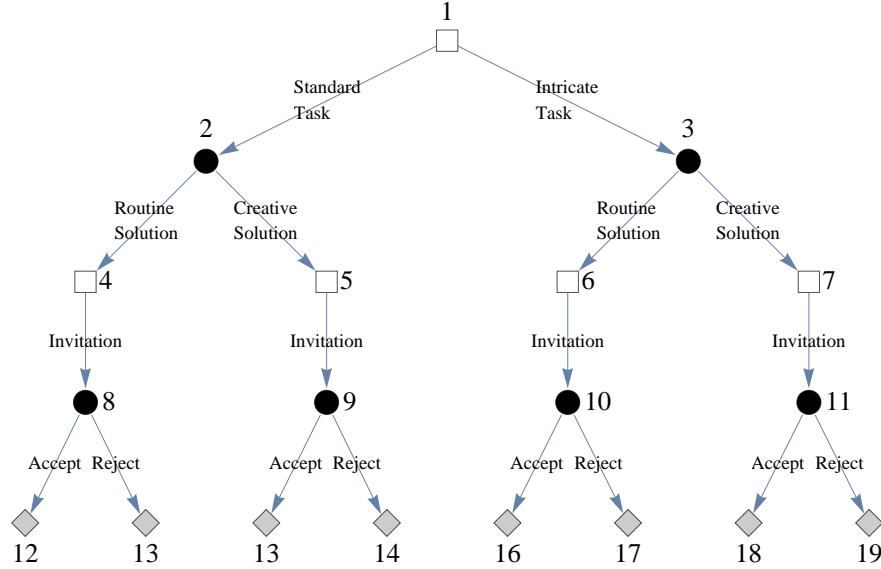
FIGURE 1: *Game tree. D_A contains the nodes depicted as squares, D_B contains the nodes depicted as circles. Diamonds are the ending nodes.*

Master thesis are given by two different contributions θ_1 and θ_2.[21] The first one reflects Bob's necessity to get his Master studies finished so a personal challenge for his final dissertation is not very desirable. The second one reflects the joy for a motivated student when solving a challenge.

Third: After Bob finishes his Master thesis, Alice invites him (Inv) for a PhD collaborating in her specific project.

Fourth: Bob can accept (A) or reject (R) such offer. Bob's preferences in this stage are given by two choice-making eigentypes: τ_1 if he is open-minded and willing to work with Alice, or τ_2 if he will not trust her very specific idea and will refuse her as PhD advisor.

Description of this interaction as a TI game of maximal information:

1. $I = \{$Alice, Bob$\}$, labeled as A and B, respectively.
2. The tree of the game is shown in Figure 1, giving the decision-situations and the timing of the game. $D_A = \{1; 4\text{-}7\}$, and $D_B = \{2\text{-}3; 8\text{-}11\}$ are the decision-situations for A and B, respectively.
3. The space of types is such that player A has a unique and trivial type, while $\mathcal{T}_B = \text{Span}\big(\{\theta_1, \theta_2\}\big) = \text{Span}\big(\{\tau_1, \tau_2\}\big)$. The decomposition for the player A is trivial in the sense that there is no indeterminacy relevant to the game for this player. The player B presents two sets of eigentypes, associated to the measurements of the θ-preferences in nodes 2-3 and of the τ-preferences in nodes 8-11, with

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \sqrt{0.4} & \sqrt{0.6} \\ \sqrt{0.6} & -\sqrt{0.4} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \tag{12}$$

---

[21]We shall understand this two different aspects of Bob's personality as two orthogonal eigentypes of θ-preferences. See Definition 2.1.
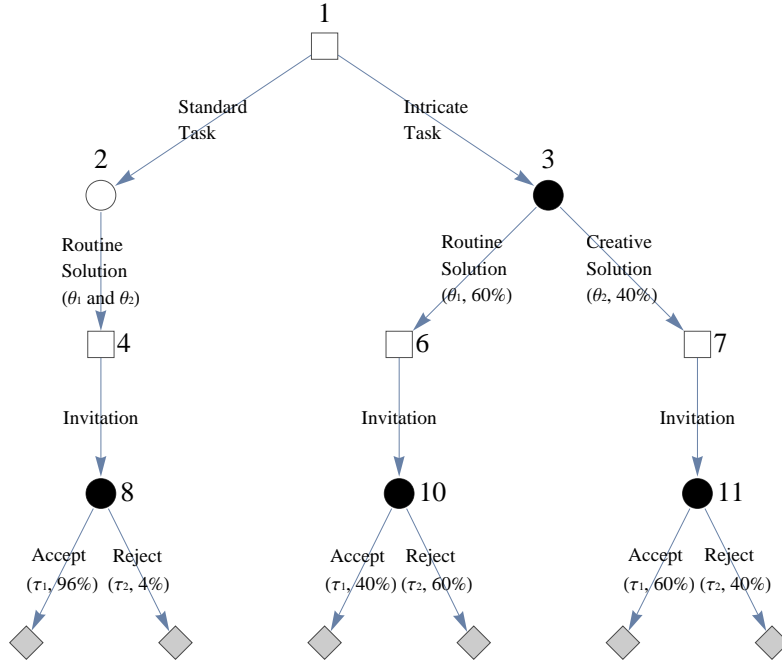
FIGURE 2: *Reduced game tree after discussion in section 2.2, showing only the actions that will be taken according to the strategic reasoning. Also the probabilities for each type being realized are written in the links. Note the differences in the probabilities for $\{\tau_1, \tau_2\}$ according to the different paths of measurements. Node 2 corresponding to Bob has been differentiated by color from node 3 to emphasize that the decision-situation 2 is associated to a trivial measurement which does not clear uncertainty in the preferences. See the discussion in the text.*

giving the correlations between the orthonormal basis.

4. The initial state for the player A is fully determined, and for the player B we have $t_B^0 = (\sqrt{0.6}, \sqrt{0.4})_\theta$, as given parameters. From (12), t_B^0 is given in coordinates of the τ-basis of eigenpreferences by $t_B^0 = (\sqrt{0.96}, \sqrt{0.04})_\tau$.

5. The available actions for each player are labeled in the Figure 1.

6. Specification of the payoffs:

   (i) The payoffs that both players will receive from their collaboration in the Master thesis are given in the following two matrices:

θ_1	S	I
R	(0; 0)	(-10; 20)
C	(-5; 0)	(-15; 70)

and

θ_2	S	I
R	(0; 0)	(5; 20)
C	(-5; 0)	(10; 70)

, (13)

representing $(u_B^{\theta_1}, u_A)$ and $(u_B^{\theta_2}, u_A)$, respectively. Alice's payoff structure represents the fact that she would prefer at this point of her collaboration with the student to give him a difficult problem. On the other hand, Bob's payoff structure reflects that for the

eigentype θ_1, receiving a standard offer is always more pleasant, as well as working in a routine basis is always more profitable, regardless of the task being the standard or the intricate one. For the eigentype θ_2, a personal challenge is more enjoyable, so receiving an intrincate task is more pleasant than receiving a standard problem. Dealing with an intricate task in the creative way is the most preferred action, but if the received task is the standard one, delivering a routine solution is also acceptable.

(ii) For doing the PhD, Bob will get a fixed amount of utility u due to the degree he earns and the scholarship he receives, regardless of who is his supervisor (the τ_i eigentypes as choice-making types have already been explained above). For Alice, who really wants to see her new idea developed, if Bob agrees (A) to work with her, she will receive $u_A(A, Inv) = 200$ or $u_A(R, Inv) = 0$ if he rejects and goes with another advisor.

Attending to the payoffs given in the matrices (13), the θ_1-type contribution to Bob's personality makes him prefer a routine (R) solution to a standard task (S) proposed by Alice as well as if it were the case that she offers the intricate one (I), while the θ_2-type contribution wills to give a routine (R) solution to a standard task (S) but a creative solution to an intricate problem (I).

From the setup of the TI game, Bob's initial type is the pure state $t_B^0 = (\sqrt{0.6}, \sqrt{0.4})_\theta$, which according to Postulate 1 reflects that Bob's personality is composed of the characteristics of θ_1 and θ_2, with weights of 60% and 40%, respectively.

Two possible paths:

1. σ_1: At node 2, after Alice offered a standard task (S), Bob will always give a routine solution (R). Because of Alice's standard proposal, there is no contradiction between the willingness of the two contributions to Bob's personality, so Bob's reaction does not clear the indeterminacy in his preferences in the sense of realizing a particular θ_i.
2. σ_2: At node 3, after Alice offered an intricate task (I), the two contributions to Bob's personality disagree on the desired action to take. At this point, the action he takes will project his preferences onto θ_1 if he goes for a routine solution (R), or onto θ_2 if he goes for a creative solution (C). See Postulate 2.

For both paths, the utilities have to be computed in expected terms since there are some non-deterministic lotteries. In the case of σ_1, there is only one move of chance: the final measurement on τ. In σ_2 there are two moves of chance: the initial measurement on θ, and final the measurement on τ. It is worth to note that in this second case, the composition of the incoming pure state for the final measurement differs depending on what was the outcome of the initial measurement. Taking these observations into account, the game tree in Figure 1 is reduced to the one in Figure 2. See the different composition of the pure states for the final measurement (recall that Alice is interested on getting Bob to accept the research project).

Utilities and TINE: Computing the utility levels $u_A[\sigma_1] = 0.96 \cdot 200 = 192$, and $u_B[\sigma_1] = u$

is straightforward. For the second path,

$$
\begin{aligned}
u_A[\sigma_2] &= p_3(\theta_1)\Big\{u_A(R,I) + p_{10}(\tau_1)u_A(A,Inv) + p_{10}(\tau_2)u_A(R,Inv)\Big\} \\
&+ p_3(\theta_2)\Big\{u_A(C,I) + p_{11}(\tau_1)u_A(A,Inv) + p_{11}(\tau_2)u_A(R,Inv)\Big\}
\end{aligned}
$$

gives $u_A[\sigma_2] = 0.6(20 + 0.4 \cdot 200 + 0.6 \cdot 0) + 0.4(70 + 0.6 \cdot 200 + 0.4 \cdot 0) = 136$, and

$$
\begin{aligned}
u_B[\sigma_2] &= p_3(\theta_1)\Big\{u_B(\theta_1,R,I) + p_{10}(\tau_1)u_B(\tau_1,A,Inv) + p_{10}(\tau_2)u_B(\tau_2,R,Inv)\Big\} \\
&+ p_3(\theta_2)\Big\{u_B(\theta_2,C,I) + p_{11}(\tau_1)u_B(\tau_1,A,Inv) + p_{11}(\tau_2)u_B(\tau_1,R,Inv)\Big\}
\end{aligned}
$$

gives $u_B[\sigma_2] = 0.6(-10 + 0.4 \cdot u + 0.6 \cdot u) + 0.4(10 + 0.6 \cdot u + 0.4 \cdot u) = u - 2$.

Then, the Type Indeterminate Nash Equilibrium of this game is such that Bob's preferred actions are Routine Solution (R) both for θ_1 and θ_2 eigentypes, when confronting a Standard Task. Thus, when Alice proposes the Standard Task (S), Bob's preferences are projected only in the last stage of the game, with the non-deterministic measurement of the choice-making types τ_1 and τ_2 with high probability of Bob accepting the PhD project, the ultimate purpose of Alice. This example can be considered as a simplified model of self-control in the sense that in this interaction, Alice restrains herself from overwhelming Bob with a difficult task when he is under the pressure of finishing his Master thesis, so that his willingness to engage in the PhD project remains intact.

Strategic manipulation: Bob's final decision, i.e., whether to accept or to reject the PhD proposal is a measurement with respect to the τ-eigentypes of preferences. As it is reflected in the Figure 2, the composition of his final preferences is affected by the presence (or absence) of a previous projection onto any of the θ_i's. The interesting feature is that this measurement will determine Bob's final preferences in the game, but it is Alice who has the power to manipulate the composition of his preferences by means of her initial choice. The sequence of measurements, if some of these are non-commuting, forces an evolution of the preferences that can be selected by the players directly with their choice of strategies, a distinctive contribution of the Type Indeterminacy model: an open door towards a theory of games with full interaction between the agents.

6. Concluding remarks

The Type Indeterminacy approach provides explanation for a wide range of behavioral anomalies in simple decision-making, and after the present proposal of extending the formalism to games with TI players, we conjecture that further development of equilibrium concepts for TI games can provide important insights in the study of behavioral patterns in interactive situations. The study of the dynamics of the Type Indeterminate interactions can contribute to the fundamental discussion suggested by Daniel Kahneman: "Is the intuitively attractive judgment or course of action in conflict with a rule that the agent would endorse?" (Kahneman, 2003).

Here, the eigentypes represent the associative or intuitive level of the reasoning process, while the notion of the type as a linear superposition of these eigenpreferences subject to the measurement process implies the existence of a rule-governed level of reasoning contained in the functional form adopted for the definition of the utility along the path.

The approach we have presented in this paper departs from the standard assumption of reference-independence, because the final states are no longer the only carriers of utility. It retains the tractability of expected utility theory however, and respects the paradigm of utility maximization at all levels of reasoning.

The TI model challenges the claim by Kahneman and Thaler (2006) that when considering some "virtuous choices that people make may involve a lack of empathy for the future self who will have to live with the choice [...] it is unlikely that these conflicting choices are both utility maximizing." The concept of Type Indeterminate Nash Equilibrium builds on the best-replies of the (potential) eigentypes of the players in a way similar to an agent-form game, and at the same time, the TINE reflects optimization at the level of the player through the maximization of the overall expected utility along the path of the game.

One of the first extensions which can be considered in this research program is the introduction of time-dependent preferences. When selecting the most preferred action, each eigentype is responsible not only of what happens in the present interacting situation, but also of how such outcome conditions the future composition of the type of the player as a consequence of the measurement process. In the example that we have illustrated, we considered both situations (the Masther Thesis and the future collaboration) to be equally weighted as a simplifying assumption for the ease of exposition. Nevertheless, the introduction of the appropriate terms to represent constant discount rates, hyperbolic discounting as well as different kinds of myopia (the ability of each eigentype to forecast, for example, only a few number of steps ahead) can be easily contained in the formulation, enriching the effects that can be described. See, *e.g.*, Lambert-Mogiliansky and Busemeyer (2012), where they study different behavioral aspects related to the concept of self-control within the framework of one TI decision-maker with temporal discounting. A classical discussion of several models of time-dependent preferences can be found in Loewenstein (2008).

Despite of some conceptual differences, the TI approach shares one of the nicest characteristics of the Projective Expected Utility model introduced by La Mura (2009). The cashed utility of an action can also be written as a bilinear form, and therefore we can introduce very naturally some interferences (penalties or rewards) due to the agents experiencing the superposition of some (confronting or reinforcing) contributions by some of the mutually exclusive eigentypes. Since the eigentypes are orthonormal by definition, this makes it very easy to think about contradictory contributions in the personality and preferences of the players.

The extension from pure states to the formulation in terms of density matrices has been excluded from this paper to avoid computational complexity, in order to keep the discussion as fundamental and formal as possible according to our best.

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